Discretization of vector bundles and rough Laplacian

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Abstract

Let $\mathcal{M}(m,\kappa,r_0)$ be the set of all compact connected m-dimensional manifolds (M,g) such that $Ricci(M,g)\geq -(m-1)\kappa g$ and $Inj(M,g)\geq r_0>0$. Let $\mathcal{E}(n,k_1,k_2)$ be the set of all Riemannian vector bundles (E,∇) of real rank n with $|R^E|\leq k_1$ and $|d^*R^E|\leq k_2$. For any vector bundle $E\in\mathcal{E}(n,k_1,k_2)$ with harmonic curvature or with complex rank one, over any $M\in\mathcal{M}(m,\kappa,r_0)$ and for any discretization X of M of mesh $0<\varepsilon\leq\frac{1}{20}r_0$, we construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E and M such that we can compare uniformly the spectrum of the rough Laplacian $\overline{\Delta}$ associated to the connection of E and the spectrum of Δ_A+V. We show that there exist constants c, $c'>0$ depending only on the parameters of $\mathcal{M}(m,\kappa,r_0)$ and $\mathcal{E}(n,k_1,k_2)$ such that $c'\lambda_k(X,A,V)\leq\lambda_k(E)\leq c\lambda_k(X,A,V)$, where $\lambda_k(\cdot)$ denotes the k^{th} eigenvalue of the considered operators ($k\leq n|X|$). For flat vector bundles, we show that the potential is zero, Δ_A turns out to be a discrete magnetic Laplacian and we relate $\lambda_1(E)$ to the holonomy of E.

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1 Introduction

In [22], we have shown that for a family of compact connected manifolds $\mathcal{M}(m,\kappa,r_0)$ with injectivity radius and Ricci curvature bounded below (i.e. $(M,g)\in\mathcal{M}(m,\kappa,r_0)$ if M is a compact connected m-dimensional Riemannian manifold with $Ricci(M,g)\geq -(m-1)\kappa g$ and $Inj(M,g)\geq r_0$), we can compare uniformly the spectrum of the Laplacian acting on functions with

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the spectrum of the combinatorial Laplacian acting on a graph with fixed mesh constructed on the manifolds. Indeed, we show that there exist positive constants c, c' depending on the parameters of the problem such that for any $M \in \mathcal{M}(m, \kappa, r_0)$ and any discretization X of M (with mesh $\varepsilon < \frac{1}{2}r_0$), the following holds

$$c'\lambda_k(X) \leq \lambda_k(M) \leq c\lambda_k(X) \tag{1.1}$$

for $k < |X|$, where $\lambda_k(\cdot)$ stands for the k^{th} eigenvalue of the considered Laplacian. This result generalizes in a natural way different works like [5], [6], [9] and [19] that were motivated either by the study of the relation between the fundamental group of a manifold and the spectrum of its finite coverings ([5], [6]) or by the relation between the spectrum of a manifold and its Cheeger isoperimetric constant ([9]) or by the existence of harmonic functions ([19]). More generally, the aim of the discretization is to have an understanding of the spectrum (a global invariant on the manifold) with a minimum of informations about the local geometry of the manifold.

Of course, the problem is interesting for differential operators other than the Laplacian and we may address the following question: does the same kind of comparison hold for other geometric differential operators such that the Laplacian acting on p-forms or the Dirac operator? Most of these operators may be expressed in terms of a connection Laplacian added with a curvature term. In this article, we investigate the case of such a connection (or rough) Laplacian $\overline{\Delta}$ associated to a connection ∇ on a vector bundle. More precisely, the purpose is to establish a uniform comparison of spectra between rough Laplacians on vector bundles and twisted Laplacians on graphs that generalize combinatorial or discrete magnetic Laplacians. The Riemannian vector bundles we are interested in have curvature and exterior coderivative of curvature bounded i.e. we study Riemannian vector bundles E with fiber of real rank n such that $|R^E| \leq k_1$ and $|d^*R^E| \leq k_2$ (denote by $\mathcal{E}(n, k_1, k_2)$ the set of such vector bundles). The main result (Theorem 3.1) states that there exist positive constants c, c' (depending only on the given parameters) such that for any vector bundle $E \in \mathcal{E}(n, k_1, k_2)$ over any $M \in \mathcal{M}(m, \kappa, r_0)$ satisfying one of the following assumptions

I) the curvature of E is harmonic i.e. $d^*R^E = 0$,

II) E is of complex (or quaternionic) rank one

and for any discretization X of E, we can construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E such that

$$c'\lambda_k(X, A, V) \leq \lambda_k(E) \leq c\lambda_k(X, A, V) \tag{1.2}$$

for any $k \leq n|X|$, where $\lambda_k(E)$ denotes the k^{th} eigenvalue of the rough Laplacian $\overline{\Delta}$ and $\lambda_k(X, A, V)$ the k^{th} eigenvalue of $\Delta_A + V$.

The case of flat vector bundles is especially enlightening. Indeed, if E is flat, we show that the potential V is zero and that Δ_A is a discrete magnetic Laplacian. This particular case shows how the construction of Δ_A is strongly related to the holonomy of E. This fact is emphasized by Theorem 4.1 which relates the holonomy (in the sense of [2]) to the first eigenvalue of Δ_A and therefore of $\overline{\Delta}$. In order to understand the problem of non-flat vector bundles, go back to the case of functions. Recall that for functions we needed to establish correspondances between functions on the manifold and functions on the graph. To that aim and in particular to associate smooth functions to functions on the graph, we had to extend locally such a function in a constant way and then smooth it (with a partition of unity). The question of extending locally is a central problem for the case of vector bundles. It turns out that extending by parallel transport is really efficient for flat vector bundles as it produces parallel sections. But, as soon as the curvature comes in, parallel transport is not convenient anymore and we need to construct a finer way to extend locally a section. In fact, the obstruction to extend in a parallel manner is double: the holonomy plays the role of a global obstruction to extend as parallel as possible and locally the curvature plays the same role. The twisted Laplacian will precisely render the holonomy of the vector bundle, while the potential will take into account the local non-flat geometry.

The paper is organized as follows. In Section 2, we introduce the notations, we define the general notion of twisted Laplacian on a graph and recall the main properties of the discretization of a manifold (that will coincide with the notion of discretization of vector bundles). Section 3 is devoted to the proof of the main result (Theorem 3.1). The main difficulty is to construct a suitable twisted Laplacian (see Section 3.1). From a geometric point of view, the problem is the dependence on the local geometry of the Laplacian and the potential to have enough informations to estimate the spectrum of the vector bundle. Technically, we need fine analysis on vector bundles like Sobolev inequalities for sections to achieve the construction. The particular case of flat vector bundles can be kept in mind as the ground example during the reading. In this case, the proofs can be done easier (we can avoid the technical tools described in Section 3.1). Nevertheless, this case already contains the essential information for Δ_A as it shows how the holonomy is related to Δ_A (see Section 4). For non-flat vector bundles, Δ_A does not suffice anymore to control the rough Laplacian, so that we have to introduce a potential V which takes care of the curvature locally. The generalization of

the flat case is then done for two different cases (see assumptions I) and II)), for rank one vector bundles and for vector bundles with harmonic curvature. These two cases are really of different nature. This appears all along Section 3 and this begins with the construction of $\Delta_A + V$ (in Section 3.2) which differs according to the assumptions I) or II). In Section 4, we establish the relationship between the holonomy and the first eigenvalue of the rough Laplacian for flat vector bundles. The part of Theorem 4.1 that bounds from below the first eigenvalue in terms of the holonomy can be generalized easily to vector bundles with harmonic curvature. But this will not be done here. This result is in fact due to Ballmann, Brüning and Carron in a more general setting (see [2]). Finally, we collect some more technical proofs in the appendix to make easier the reading, even if the results are not of minor importance for the paper.

2 Settings

2.1 Rough Laplacian

In this section, we recall basic facts on the rough Laplacian (for a general reference see [3], [24] or [25] for instance). Let (M, g) be a compact connected m-dimensional Riemannian manifold without boundary and with volume form denoted by dV. Moreover, let (E, ∇) be a Riemannian vector bundle with n-dimensional fiber over M i.e. E is a vector bundle over M endowed with a smooth metric $\langle \cdot, \cdot \rangle$ and a compatible connection ∇. On the set $\Gamma(E)$ of smooth sections of E, denote by $(\cdot, \cdot)$ the L^2-inner product endowed by $\langle \cdot, \cdot \rangle$ and g. Recall that the connection extends to p-tensors on M with values in E and that we define ∇^* to be the adjoint of ∇ with respect to the L^2-inner product. The rough Laplacian (or connection Laplacian) acting on $\Gamma(E)$ is then defined by $\overline{\Delta} = \nabla^* \nabla$. The spectrum of $\overline{\Delta}$ is discrete and non-negative and will be denoted

$$Spec(E) = \{\lambda_1(E) \leq \lambda_2(E) \leq \ldots \leq \lambda_k(E) \leq \ldots\}.$$

The Rayleigh quotient of a non-zero section s is defined by $R(s) = \frac{\|\nabla s\|^2}{\|s\|^2}$, where $\| \cdot \|$ denotes the L^2-norm associated to the L^2-inner product defined above. Later we will need the following variational characterizations of $Spec(E)$ known as min-max and max-min theorems. For any $k \geq 1$,

$$\begin{aligned} \lambda_k(E) &= \min_{\Omega^k} \max\{R(s) : s \in \Omega^k \setminus \{0\}\} \\ &= \max_{\Omega^{k-1}} \min\{R(s) : s \in \Omega^{k-1} \setminus \{0\},\ s \perp \Omega^{k-1} \text{ w.r.t } (\cdot, \cdot)\} \end{aligned}$$

where Ω^k (resp. Ω^{k-1}) ranges over all k-dimensional (resp. $(k-1)$-dim.) subspaces of $\Gamma(E)$.

2.2 Twisted Laplacian

Let $\Gamma = (X, E(X))$ be a finite connected graph endowed with the path metric. For $p \in X$ denote by $N(p)$ the set of vertices at distance 1 from p and by $m(p)$ the number of such vertices. In order to generalize the combinatorial Laplacian (see [21] for a definition) and the discrete magnetic Laplacian (see [23] for a definition), let us consider the set of functions on X with values in $\mathbb{R}^n$ i.e. $\mathcal{F}(X) = \{f : X \to \mathbb{R}^n\}$, provided with the inner product $(f, g) = \sum_{p \in X} f(p) \cdot g(p)$, where $\cdot$ denotes the Euclidean inner product of $\mathbb{R}^n$.

Definition 2.1 *For any $p \in X$ and $q \in N(p)$ assume that $A(p,q) : \mathbb{R}^n \to \mathbb{R}^n$ is a given linear transformation. The* ***twisted Laplacian*** *associated to A is the operator $\Delta_A : \mathcal{F}(X) \to \mathcal{F}(X)$ defined by*

$$\Delta_A f(p) = \frac{1}{2} \sum_{q \in N(p)} \left(\mathbb{I} + A^t(p,q)A(p,q)\right) f(p) - \left(A(q,p) + A^t(p,q)\right) f(q).$$

Remark 2.2 *If for any p, q, the operator $A(p,q)$ is the identity, then Δ_A is the combinatorial Laplacian.*

Remark 2.3 *If $A(p,q)$ belongs to $O(n)$ and $A^t(p,q) = A(q,p)$, then $\Delta_A f(p) = m(p) f(p) - \sum_{q \in N(p)} A(q,p) f(q)$. In this case the twisted Laplacian is usually called* ***discrete magnetic Laplacian*** *or Laplacian associated to the Harper operator A.*

Let us introduce the space of functions $\mathcal{F}(X \times X) = \{F : X \times X \to \mathbb{R}^n\}$ and provide it with the inner product given by $(F, G) = \frac{1}{2} \sum\limits_{p \in X} \sum\limits_{q \in X} F(p,q) \cdot G(p,q)$.

Lemma 2.4 *Let $A(p,q)$ be as in Definition 2.1 and Δ_A the twisted Laplacian associated to A. Let $D_A : \mathcal{F}(X) \to \mathcal{F}(X \times X)$ be defined by*

$$D_A f(p,q) = \begin{cases} f(q) - A(p,q) f(p) & \textit{if } p \in N(q), \\ 0 & \textit{otherwise.} \end{cases}$$

Then, for any f,$g \in \mathcal{F}(X)$, we have $(\Delta_A f, g) = (D_A f, D_A g)$.

Proof: let $f,\, g \in \mathcal{F}(X)$. Then, we have

$$\begin{aligned}
(\Delta_A f, g) &= \frac{1}{2}\sum_{p\in X}\sum_{q\in N(p)} (f(p) - A(q,p)f(q)) \cdot g(p) \\
&\quad -\frac{1}{2}\sum_{p\in X}\sum_{q\in N(p)} (f(q) - A(p,q)f(p)) \cdot A(p,q)g(p) \\
&= \frac{1}{2}\sum_{p\in X}\sum_{q\in X} D_A f(q,p) \cdot g(p) + \frac{1}{2}\sum_{p\in X}\sum_{q\in X} D_A f(p,q) \cdot D_A g(p,q) \\
&\quad -\frac{1}{2}\sum_{p\in X}\sum_{q\in X} D_A f(p,q) \cdot g(q) \ = \ (D_A f, D_A g). \qquad \square
\end{aligned}$$

A direct consequence of this lemma is that Δ_A is symmetric and non-negative, so it admits a non-negative spectrum. If $V : \mathcal{F}(X) \to \mathcal{F}(X)$ is a non-negative potential, then the spectrum of $\Delta_A + V$ is characterized by min-max theorem as follows

$$\forall 1 \leq k \leq n|X|, \qquad \lambda_k(X, A, V) = \min_{W^k} \max\{R(f) : f \in W^k \setminus \{0\}\}$$

where W^k ranges over all k-dimensional vector subspaces of $\mathcal{F}(X)$ and $R(f)$ is the Rayleigh quotient of f defined by $R(f) = \frac{\|D_A f\|^2 + (Vf,f)}{\|f\|^2}$.

2.3 Discretization of vector bundles

In this section, we define the notion of discretization of a vector bundle.

Definition 2.5 *Let (E, ∇) be a Riemannian vector bundle over (M, g) a compact connected Riemannian manifold with $\partial M = \emptyset$. An* ***ε-discretization*** *of E is a discretization of M of mesh $\varepsilon > 0$.*

The discretization of a manifold (of mesh ε) is defined as in [10] (Section V.3.2). Let us recall the definition and the properties of such a discretization. Let (M, g) be a compact connected m-dimensional Riemannian manifold. A discretization of M, of mesh $\varepsilon > 0$, is a maximal ε-separated subset X of M provided with a graph structure given by the sets $N(p) = \{q \in X \mid 0 < d(p,q) < 3\varepsilon\}$, for any $p \in X$. In other words, X is such that for any distinct $p,\, q \in X$, $d(p,q) \geq \varepsilon$ and $\bigcup_{p\in X} B(p,\varepsilon) = M$. Moreover, pq is an edge if and only if $0 < d(p,q) < 3\varepsilon$. Denote by $m(p)$ the number of elements of $N(p)$.

Remark 2.6 *Let us remark that if $B(p,\rho)$ is a ball in M with radius $\rho < \frac{1}{2} Inj(M)$, then the volume $V(p,\rho)$ of the ball $B(p,\rho)$ is bounded below by a*

constant depending only on ρ and m (this is Croke's Inequality, see for instance in [10] p.136). Moreover, if M has Ricci curvature bounded below by $-(m-1)\kappa$ then the volume of a ball of radius R is bounded above by a constant depending only on m, κ and R (this follows from Bishop's comparison theorem, see for instance [10], p.126). These bounds will be used frequently in the sequel.

Choose ε smaller than $\frac{1}{2}Inj(M)$. Denote by $\kappa \geq 0$ a constant such that $Ricci(M,g) \geq -(m-1)\kappa g$. Then, using Remark 2.6 we can show that $m(p)$ is bounded above by a constant ν_X depending only on m, κ and ε and that $\frac{1}{V_{-\kappa}(\varepsilon)}Vol(M) \leq |X| \leq \frac{2^m}{\varepsilon^m c(m)}Vol(M)$, where $V_{-\kappa}(\varepsilon)$ denotes the volume of the ball of radius ε in the simply connected space of constant sectional curvature $-\kappa$ and of dimension m.

3 Spectra comparison for rough Laplacian and twisted Laplacian

In this section, we will establish the comparison between the spectra of the rough Laplacian and a twisted Laplacian. Let us state the main result.

Theorem 3.1 *Let m, n be positive integers, κ, k_1, $k_2 \geq 0$ and $r_0 \geq 20\varepsilon > 0$. There exist positive constants c, c' depending only on m, n, κ, k_1, k_2 and ε such that for any $M \in \mathcal{M}(m,\kappa,r_0)$, any vector bundle $E \in \mathcal{E}(n,k_1,k_2)$ over M satisfying one of the following condition*

I) the curvature of E is harmonic i.e. $d^*R^E = 0$,

II) E is of complex (or quaternionic) rank one

and for any ε-discretization X of E, we can construct a canonical twisted Laplacian Δ_A and a potential V depending only on the local geometry of E such that, for $1 \leq k \leq n|X|$

$$c'\lambda_k(X,A,V) \leq \lambda_k(E) \leq c\lambda_k(X,A,V).$$

In particular, if the vector bundle is flat, the potential is zero and Δ_A is a discrete magnetic Laplacian.

Roughly speaking, the basic idea of the proof is the same as to prove the theorem of comparison of spectra between the Laplacian acting on functions and the combinatorial Laplacian ([22], Theorem 3.7). But a first fundamental difference between the functions and the vector bundles cases is the construction of the twisted Laplacian. Indeed, in [22] the combinatorial Laplacian

appearing in Theorem 3.7 is canonically associated to the graph that discretizes the manifold. For vector bundles, such a canonical Laplacian on graphs does not obviously exist. Hence, a first step of the proof consists in constructing a suitable twisted Laplacian Δ_A and a potential V (Section 3.2) that will depend only on the local geometry. The construction of $\Delta_A + V$ differs according to the assumptions I) and II). We will work with balls centered on X and for both cases the construction of Δ_A relies essentially on changes of bases from a ball to a neighboring ball, but for vector bundles satisfying II) the definition of Δ_A is slightly harder. A more significant difference is the construction of the potential V. For rank one vector bundles, V involves only the first eigenvalue of balls (with Neumann boundary condition), while in the other case, we will distinguish "small" eigenvalues of balls from "large" eigenvalues. In rank one vector bundles the n first eigenvalues (of such a ball) are the same and correspond to the minimum of the energy, so that it will make easier the estimating of V.

After defining the twisted Laplacian and the potential, we follow the same way of proof as for the case of functions, but the underlying analysis is much more difficult. For instance, we need to establish some Sobolev inequalities for sections that requires fine tools of analysis as Moser's iteration and Sobolev inequalities for functions (Lemma A.1 in Appendix). The definition of the smoothing operator $\mathcal{S}$ and the discretizing operator $\mathcal{D}$ generalizes in some sense the similar operators defined by Chavel in [10] (Sections VI.5.1 and VI.5.2). Similarly, we establish norms estimations for these operators $\mathcal{S}$ and $\mathcal{D}$ (Propositions 3.18 and 3.21) in order to compare Rayleigh quotients of sections with Rayleigh quotients of functions on the discretization. Then, min-max theorem leads to the result for "small" eigenvalues. It suffices moreover to have upper bounds on the respective spectra (Lemma 3.23) to compare "large" eigenvalues and conclude the proof of Theorem 3.1 (Section 3.6).

3.1 Local extension

In this section we define a way to extend a section as parallel as possible. In the case of flat vector bundles parallel transport is the suitable tool, because of the lemma below. Let $\tau_{x,p}$ denotes the parallel transport from E_p to E_x along the minimizing geodesic joining p to x (for $d(p,x) < \frac{1}{2}Inj(M)$).

Lemma 3.2 *Let (E, ∇) be a flat Riemannian vector bundle over a Riemannian manifold (M, g). Let $p \in M$ and $B(p, r)$ the ball centered at p of radius $r < \frac{1}{2}Inj(M)$. Then for any $v \in E_p$, the section σ over $B(p, r)$ defined by $\sigma(x) = \tau_{x,p}v$ is parallel.*

Proof: see [12] Section 2.2.1. $\square$

In the non-flat case, extending by parallel transport is not strong enough for our purpose, because we need to control the covariant derivative of such extended sections. More precisely, we want to extend in an energy minimizing way. This means that we have to take into account local small eigenvalues. Hence, we introduce eigensections of the Neumann problem on balls which give an obstruction to extension in a parallel way. Such eigensections on balls associated to small eigenvalues are almost parallel (Lemma 3.3) and will provide a good way to extend sections. Nevertheless, it may happen that there are no (or only a few) small eigensections on a ball. In this case, parallel transport will be good enough to extend as we will see.

Lemma 3.3 *Let $(E, \nabla) \in \mathcal{E}(n, k_1, k_2)$ over $(M, g) \in \mathcal{M}(m, \kappa, r_0)$. For $0 < r < \frac{1}{2}r_0$ and $p \in M$, let $\sigma : B(p, r) \to E$ be a section such that $\overline{\Delta}\sigma = \lambda\sigma$ for a constant $\lambda \geq 0$. Let $0 < \theta < 1$. Then there exist $0 < c(m) \leq s \leq 1$ and c, $c' > 0$ depending on an upper bound for λ and on m, n, κ, r, k_1, k_2 and θ such that*

$$\begin{aligned} \|\sigma\|_{\infty,\theta r} &\leq c\|\sigma\|_{2,r}, \\ \|\nabla\sigma\|_{\infty,\theta r} &\leq c'\|\nabla\sigma\|_{2,r}^{s} \end{aligned}$$

where $\|\cdot\|_{q,\rho}$ denotes the L^q-norm on the ball centered at p of radius ρ (c' depends on $c\|\sigma\|_{2,r}$ too).

Moreover, there exists $c'' > 0$ depending on c, c' and r such that

$$|\sigma(x) - \tau_{x,p}\sigma(p)| \leq c''\|\nabla\sigma\|_{2,r}^{s}$$

for all $x \in B(p, \theta r)$. If $k_2 = 0$ i.e. if E is of harmonic curvature, then $s = 1$ in the previous inequalities.

Proof: the idea is to use a Moser iteration to prove the statement. The more technical part of the argument is carried out in the appendix (see Lemma A.1). In order to use Lemma A.1, let $\delta > 0$ and $u_\delta : B(p, r) \to \mathbb{R}$ defined by $u_\delta = \sqrt{|\sigma|^2 + \delta}$. Then in one hand $\Delta(u_\delta^2) = 2u_\delta\Delta u_\delta - 2|du_\delta|^2$ and in the other hand $\Delta(u_\delta^2) = 2\langle\sigma, \overline{\Delta}\sigma\rangle - 2|\nabla\sigma|^2$ which implies that

$$u_\delta\Delta u_\delta \leq \langle\sigma, \overline{\Delta}\sigma\ \rangle = \lambda|\sigma|^2 \leq \lambda u_\delta^2.$$

We can then apply Lemma A.1 to u_δ and we get that $\|u_\delta\|_{\infty,\theta r} \leq c\|u_\delta\|_{2,r}$. Then let $\delta \to 0$ to obtain the first claim.

For the second inequality, let $\delta > 0$ and $v_\delta : B(p, r) \to \mathbb{R}$ defined by $v_\delta(x) = \sqrt{|\nabla\sigma(x)|^2 + \delta}$. Then

$$\Delta(v_\delta^2) = 2v_\delta\Delta v_\delta - 2|dv_\delta|^2 = 2\langle\nabla\sigma, \overline{\Delta}(\nabla\sigma)\rangle - 2|\nabla\nabla\sigma|^2.$$

But we have that $|\nabla\nabla\sigma|^2 - |dv_\delta|^2 \geq 0$ and therefore

$$v_\delta \Delta v_\delta \leq \langle \nabla\sigma, \overline{\Delta}(\nabla\sigma)\rangle = \langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle + \lambda|\nabla\sigma|^2.$$

By a commuting argument (see [1], Lemma 2.3) we have for a local orthonormal frame $\{X_i\}_{i=1,\ldots,m}$ of M

$$\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle =$$
$$\lambda|\nabla\sigma|^2 - \langle \nabla_{Ric(\cdot)}\sigma, \nabla\sigma\rangle - 2\sum_{i=1}^{m}\langle R^E(X_i,\cdot)\nabla_{X_i}\sigma, \nabla\sigma\rangle + \langle (d^*R^E)\sigma, \nabla\sigma\rangle$$

and as $Ricci(M,g) \geq -(m-1)\kappa g$, $|R^E| \leq k_1$ and $|d^*R^E| \leq k_2$ we then get

$$\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle \leq \left(\lambda + (m-1)\kappa + 2n^2k_1\right)|\nabla\sigma|^2 + n^2k_2|\sigma||\nabla\sigma|.$$

By the first part of the proof, we obtain that on $B(p,\theta r)$

$$\langle \nabla\sigma, \overline{\Delta}(\nabla\sigma) - \nabla(\overline{\Delta}\sigma)\rangle \leq$$
$$\left(\lambda + (m-1)\kappa + 2n^2k_1\right)|\nabla\sigma|^2 + n^2k_2c\|\sigma\|_{2,r}|\nabla\sigma|$$

and this implies (on $B(p,\theta r)$)

$$\Delta v_\delta \leq \left(\lambda + (m-1)\kappa + 2n^2k_1\right)v_\delta + n^2k_2c\|\sigma\|_{2,r}.$$

If $\theta' < \theta$ we can apply Lemma A.1 to v_δ and let $\delta \to 0$ to obtain

$$\|\nabla\sigma\|_{\infty,\theta' r} \leq c'\|\nabla\sigma\|_{2,\theta r}^s \leq c'\|\nabla\sigma\|_{2,r}^s. \tag{3.1}$$

Note that if $k_2 = 0$, then $s = 1$ and c' does not depend on $c\|\sigma\|_{2,r}$. The two first inequalities in the statement are then true for any θ' such that $0 < \theta' < \theta < 1$. So rename θ' by θ to obtain the statement.

Finally, recall that if γ is the minimizing geodesic joining p to $x \in B(p,\theta r)$ of length l ($< \theta r$), then $|\sigma(x) - \tau_{x,p}\sigma(p)| \leq \int_0^l \left|\nabla_{\dot\gamma(t)}\sigma(\gamma(t))\right| dt \leq l\|\nabla\sigma\|_{\infty,\theta r}$. Using (3.1) leads to the result. □

From now on, let $E \in \mathcal{E}(n,k_1,k_2)$ over $M \in \mathcal{M}(m,\kappa,r_0)$ and fix $\varepsilon \leq \frac{1}{20}r_0$. Let X be an ε-discretization of E. Let $\sigma_k^p : B(p,10\varepsilon) \to E$ be the eigensection associated to the k^{th} eigenvalue $\lambda_k(p)$ of $\overline{\Delta}$ on $B(p,10\varepsilon)$ with Neumann boundary condition such that $\int_{B(p,10\varepsilon)}\langle\sigma_k^p,\sigma_l^p\rangle dV = \delta_{kl}V(p,10\varepsilon)$.

Remark 3.4 *If E is flat $\lambda_1(p) = \ldots = \lambda_n(p) = 0$ and the σ_k^p's give a local orthonormal frame over $B(p,10\varepsilon)$.*

Remark 3.5 *If $n = 2$ (resp. $n = 4$) and E is of complex (resp. quaternionic) rank one, then $\lambda_1(p) = \ldots = \lambda_n(p)$. Indeed, the section $i\sigma_1^p$ (resp. $i\sigma_1^p$, $j\sigma_1^p$, $k\sigma_1^p$ where i, j, k are the quaternions with $i^2 = j^2 = k^2 = -1$) satisfies $\nabla(i\sigma_1^p) = i\nabla\sigma_1^p$ which implies that $i\sigma_1^p$ is a $\lambda_1(p)$-eigensection orthogonal to σ_1^p. Hence, we can choose σ_k^p such that for any x in $B(p, 10\varepsilon)$, $\langle\sigma_k^p(x), \sigma_l^p(x)\rangle = 0$ for any $1 \le k \le n$, $1 \le l \le n$, $k \ne l$.*

Lemma 3.6 *Let $0 \le \alpha < \frac{1}{n+1}$. There exists $\delta > 0$ depending only on α, m, n, k_1, k_2, κ, ε such that if $\lambda_k(p) \le \delta$ then $\forall\ 1 \le i, j \le k$ and $\forall x \in B(p, 8\varepsilon)$ $|\langle\sigma_i^p(x), \sigma_j^p(x)\rangle - \delta_{ij}| \le \alpha$. In particular, if $\lambda_k(p) \le \delta$, then $\{\sigma_1^p(x), \ldots, \sigma_k^p(x)\}$ spans a k-dimensional vector subspace of E_x, for any $x \in B(p, 8\varepsilon)$.*

To prove this lemma, let us recall a basic fact of linear algebra (the proof of the fact is left to the reader). Let V be an n-dimensional vector space provided with an inner product $\langle\cdot,\cdot\rangle$. If $\{v_1, \ldots, v_n\} \subseteq V$ is such that $|\langle v_i, v_j\rangle - \delta_{ij}| \le \alpha < \frac{1}{n+1}$, then $\{v_1, \ldots, v_n\}$ is a basis of V. Moreover for any $v = \sum_{i=1}^n a_i v_i$, we have $(1-\alpha(n+1))\sum_{i=1}^n a_i^2 \le \|v\|^2 \le (1+\alpha(n+1))\sum_{i=1}^n a_i^2$. Such a basis will be referred as an **almost orthonormal basis**.

Proof of Lemma 3.6: let $f_{ij}(x) = \langle\sigma_i^p(x), \sigma_j^p(x)\rangle$ and denote by m_{ij} its mean over $B(p, 10\varepsilon)$, then

$$m_{ij} = \frac{1}{V(p, 10\varepsilon)} \int_{B(p,10\varepsilon)} f_{ij} dV = \delta_{ij}.$$

A result of Kanai ensuring the existence of $c_K > 0$ depending only on ε and κ (see [10], Lemma VI.5.5) and the assumption $\lambda_k(p) \le \delta$ imply

$$0 \le \int_{B(p,10\varepsilon)} |f_{ij} - \delta_{ij}| dV \le c_K \int_{B(p,10\varepsilon)} |df_{ij}| dV \le c_K V(p, 10\varepsilon)\sqrt{\delta}. \quad (3.2)$$

Moreover,

$$\begin{aligned} \inf_{x \in B(p,\frac{\varepsilon}{2})} \{|f_{ij}(x) - \delta_{ij}|\} V\left(p, \frac{\varepsilon}{2}\right) &\le \int_{B(p,\frac{\varepsilon}{2})} |f_{ij}(x) - \delta_{ij}| dV(x) \\ &\le c_K V(p, 10\varepsilon)\sqrt{\delta}. \quad (3.3) \end{aligned}$$

The last inequality follows from (3.2). Hence (3.3) implies that there exists $p' \in M$, $d(p, p') \le \frac{\varepsilon}{2}$, such that

$$|\langle\sigma_i^p(p'), \sigma_j^p(p')\rangle - \delta_{ij}| \le 2c_K \frac{V(p, 10\varepsilon)}{V(p, \frac{\varepsilon}{2})}\sqrt{\delta} \le c\sqrt{\delta}.$$

We conclude then as follows

$$
\begin{aligned}
&\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\delta_{ij}\right|\le\\
&\qquad\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\tau_{x,p'}\sigma_i^p(p'),\tau_{x,p'}\sigma_j^p(p')\rangle\right|+\left|\langle\sigma_i^p(p'),\sigma_j^p(p')\rangle-\delta_{ij}\right|\\
&\qquad\qquad\le\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\tau_{x,p'}\sigma_i^p(p'),\tau_{x,p'}\sigma_j^p(p')\rangle\right|+c\sqrt{\delta}. \quad (3.4)
\end{aligned}
$$

For any $x\in B(p,8\varepsilon)$ the minimizing geodesic $\overline{xp'}$ stays in $B(p,9\varepsilon)$, so we can write

$$
\begin{aligned}
\left|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\sigma_i^p(p'),\sigma_j^p(p')\rangle\right| &\le 9\varepsilon\|d\langle\sigma_i^p,\sigma_j^p\rangle\|_{\infty,9\varepsilon}\\
&\le 9\varepsilon\left(\|\nabla\sigma_i^p\|_{\infty,9\varepsilon}\|\sigma_j^p\|_{\infty,9\varepsilon}+\|\sigma_i^p\|_{\infty,9\varepsilon}\|\nabla\sigma_j^p\|_{\infty,9\varepsilon}\right)\\
&\le 9\varepsilon c'\left(\|\nabla\sigma_i^p\|_{2,10\varepsilon}^s\|\sigma_j^p\|_{2,10\varepsilon}+\|\sigma_i^p\|_{2,10\varepsilon}\|\nabla\sigma_j^p\|_{2,10\varepsilon}^s\right)
\end{aligned}
$$

where we used Lemma 3.3 in the last inequality. By definition of the σ_i^p's and by assumption on $\lambda_i(p)$ we get

$$|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\langle\sigma_i^p(p'),\sigma_j^p(p')\rangle|\le c''\sqrt{\delta^s}. \quad (3.5)$$

Finally, (3.4) and (3.5) imply that for a sufficiently small δ we have

$$|\langle\sigma_i^p(x),\sigma_j^p(x)\rangle-\delta_{ij}|\le\left(c\sqrt{\delta}+c''\sqrt{\delta^s}\right)\le\alpha<\frac{1}{n+1}$$

and this ends the proof. □

Definition 3.7 *Fix once and for all $0<\alpha<\frac{1}{n+1}$. Let δ be given by Lemma 3.6. For $p\in X$, define then $\boldsymbol{\mu(p)}$ as the largest integer such that $\lambda_{\mu(p)}(p)\le\delta$.*

Remark 3.8 *If the vector bundle is flat, $\mu(p)=n$, for any $p\in X$.*

For $p\in X$, we want to extend a section in a neighborhood of p as parallel as possible and taking care of local small eigenvalues as said before. So let us define the **local extension** that associates to a vector in E_p a local section over $B(p,10\varepsilon)$. Consider $E_{\mu(p)}$ the $\mu(p)$-dimensional vector subspace of E_p spanned by $\{\sigma_1^p(p),\dots,\sigma_{\mu(p)}^p(p)\}$. Let $E_{\mu(p)}^\perp$ be the orthogonal complement of $E_{\mu(p)}$ in E_p and choose $\{e_{\mu(p)+1}^p,\dots,e_n^p\}$ an orthonormal basis of $E_{\mu(p)}^\perp$. By construction, $\{e_1^p=\sigma_1^p(p),\dots,e_{\mu(p)}^p=\sigma_{\mu(p)}^p(p),e_{\mu(p)+1}^p,\dots,e_n^p\}$ is an almost orthonormal basis of E_p. We extend this basis on $B(p,10\varepsilon)$ by

$$e_i^p(x):=\begin{cases}\sigma_i^p(x) & \text{if } i\le\mu(p),\\ \tau_{x,p}e_i^p & \text{otherwise}\end{cases}$$

and we define the local extension of $v=\sum_{i=1}^n v_ie_i^p$ by $\sum_{i=1}^n v_ie_i^p(x)$.

Remark 3.9 *If E is flat, the local extension corresponds to the extension by parallel transport along radial geodesics. In this case, it suffices to choose any orthonormal basis $\{e_1^p, \dots, e_n^p\}$ of E_p and extend it radially to obtain $\{e_1^p(x), \dots, e_n^p(x)\}$.*

Lemma 3.10 *For any $x \in B(p, 8\varepsilon)$, $\{e_1^p(x), \dots, e_n^p(x)\}$ is an almost orthonormal basis of E_x.*

Proof: if $\mu(p) = 0$ the claim is clearly true. If $\mu(p) = n$ the claim follows from Lemma 3.6. Hence suppose $1 \leq \mu(p) \leq n-1$. By Lemma 3.6 $\langle e_1^p(x), \dots, e_{\mu(p)}^p(x)\rangle$ is $\mu(p)$-dimensional and as parallel translation preserves the inner product $\langle e_{\mu(p)+1}^p(x), \dots, e_n^p(x)\rangle$ is $(n - \mu(p))$-dimensional. So we have to show that there exists $c > 0$ such that

$$|\langle e_i^p(x), e_j^p(x)\rangle| \leq c < \frac{1}{n+1}\,, \qquad \forall 1 \leq i \leq \mu(p) < j \leq n.$$

Let us prove this estimate. As $e_j^p(p)$ and $\sigma_i^p(p)$ are orthogonal, we have

$$\begin{aligned} \left|\langle e_j^p(x), e_i^p(x)\rangle\right| &= \langle e_j^p(p), \tau_{p,x}\sigma_i^p(x) - \sigma_i^p(p)\rangle \\ &\leq \left|e_j^p(p)\right| \cdot |\sigma_i^p(x) - \tau_{x,p}\sigma_i^p(p)| = |\sigma_i^p(x) - \tau_{x,p}\sigma_i^p(p)|\,. \end{aligned}$$

By Lemma 3.3 $|\sigma_i^p(x) - \tau_{x,p}\sigma_i^p(p)| \leq c'\sqrt{\delta^s}$. Hence $\left|\langle e_j^p(x), e_i^p(x)\rangle\right| \leq c'\sqrt{\delta^s}$. Then, readjust δ if necessary to obtain $\left|\langle e_j^p(x), e_i^p(x)\rangle\right| \leq c < \frac{1}{n+1}$. □

Remark 3.11 *For the sequel, let δ' denote a constant, $0 < \delta' < 1$, such that $(1-\delta')\sum_{i=1}^n v_i^2 \leq \left|\sum_{i=1}^n v_i e_i^p(x)\right|^2 \leq (1+\delta')\sum_{i=1}^n v_i^2$, for any $x \in B(p, 8\varepsilon)$.*

Lemma 3.12 *There exists a positive constant c depending only on n, k_1, ε such that for any $p \in X$ and any $\mu(p) < i \leq n$, $\|\nabla e_i^p\|_{\infty, 9\varepsilon} \leq c$.*

Proof: let $x \in B(p, 9\varepsilon)$ and consider γ the minimizing geodesic from p to x of length l ($l < 9\varepsilon$) and $\{X_1 = \dot\gamma(t), X_2, \dots, X_n\}$ an orthonormal basis of E_x with $\nabla_{X_i} X_j = 0$. Then

$$|\nabla e_i^p(x)|^2 = \sum_{j=1}^n \left|\nabla_{X_j} e_i^p(x)\right|^2 \leq \sum_{j=1}^n \left(\int_0^l \left|\nabla_{\dot\gamma(t)} \nabla_{X_j} e_i^p(x)\right| dt\right)^2$$

but $|R^E(\dot\gamma(t), X_j)e_i^p| = |\nabla_{\dot\gamma(t)} \nabla_{X_j} e_i^p| \leq k_1$. Therefore $|\nabla e_i^p(x)|^2 \leq k_1^2 l^2 n$ and this concludes the proof. □

3.2 Construction of the twisted Laplacian

The construction of Δ_A differs according to the assumptions done on E. However, the basic idea is the same in all cases and relies on the fact that A has to express the holonomy. So let us consider $p,q \in X$, $p \in N(q)$ and let $x \in B(p, 8\varepsilon) \cap B(q, 8\varepsilon)$. Then define $a(p,q)_{ij}(x)$ by

$$e_j^p(x) = \sum_{i=1}^{n} a(p,q)_{ij}(x) e_i^q(x) \qquad \forall j = 1, \ldots, n$$

where e_i^p, e_j^q are defined in Section 3.1. We define $A(p,q) : \mathbb{R}^n \to \mathbb{R}^n$ on the canonical basis $\{e_1, \ldots e_n\}$ of $\mathbb{R}^n$ by $A(p,q)e_j = \sum_{i=1}^n A(p,q)_{ij} e_i$, where $A(p,q)_{ij}$ is defined as follows.

If E is of harmonic curvature then define $A(p,q)_{ij}$ by

$$A(p,q)_{ij} = a(p,q)_{ij}(q)$$

If E is of complex (or quaternionic) rank one then define $A(p,q)_{ij}$ by

$$A(p,q)_{ij} = \frac{1}{V_{pq}} \int_{B_{pq}} a(p,q)_{ij}(x) dV(x)$$

where B_{pq} is the ball centered at the mid-point of p and q of radius 5ε and V_{pq} denotes its volume. Note that $B_{pq} \supseteq B(p, 3\varepsilon) \cup B(q, 3\varepsilon)$.

Remark 3.13 *In the canonical basis of* $\mathbb{R}^n$*, we can write*

$$D_A f(p,q) = \sum_{i=1}^{n} D_A f(p,q)_i e_i = \sum_{i=1}^{n} \left(f_i(q) - \sum_{j=1}^{n} A(p,q)_{ij} f_j(p) \right) e_i$$

Remark 3.14 *If E is of harmonic curvature, we have by definition* $e_j^p(q) = \sum_{i=1}^n A(p,q)_{ij} e_i^q(q)$, $\forall j = 1, \ldots, n$.

Remark 3.15 *If E is flat, $a(p,q)_{ij}(x)$ is constant and so for $j = 1, \ldots, n$ and for any $x \in B(p, 8\varepsilon) \cap B(q, 8\varepsilon)$, $e_j^p(x) = \sum_{i=1}^n A(p,q)_{ij} e_i^q(x)$. Moreover, in this case $A(p,q)A(p,q)^t = Id$ and $A(p,q)^t = A(q,p)$. So that Δ_A is a discrete magnetic Laplacian.*

If E is of harmonic curvature let $V : \mathcal{F}(X) \to \mathcal{F}(X)$ be defined by

$$(Vf)(p) = \sum_{i \leq \mu(p)} \lambda_i(p) f_i(p) e_i + \sum_{i > \mu(p)} f_i(p) e_i.$$

If E is of complex (or quaternionic) rank one let $V : \mathcal{F}(X) \to \mathcal{F}(X)$ be defined by

$$(Vf)(p) = \left(\lambda_1(p) + \sum_{q \in N(p)} \lambda_1(q) \right) f(p). \tag{3.6}$$

Remark 3.16 *If the vector bundle is flat, then we have $V = 0$.*

3.3 Smoothing operator

Definition 3.17 *Let $\{\psi_p\}_{p \in X}$ be a partition of unity subordinate to the cover $\{B(p, 2\varepsilon)\}_{p \in X}$. Define the **smoothing operator** $\mathcal{S} : \mathcal{F}(X) \to \Gamma(E)$ by*

$$(\mathcal{S}f)(x) = \sum_{p \in X} \psi_p(x) \left(\sum_{i=1}^{n} f_i(p) e_i^p(x) \right)$$

where $f(p) = \sum_{i=1}^{n} f_i(p) e_i$.

Proposition 3.18 *There exist constants c_0, c_1, c_2 and $\Lambda > 0$ depending only on m, n, k_1, k_2, κ and ε such that*

i) $\forall f \in \mathcal{F}(X)$, $\|\mathcal{S}f\|^2 \leq c_0 \|f\|^2$,

ii) $\forall f \in \mathcal{F}(X)$, $\|\nabla(\mathcal{S}f)\|^2 \leq c_1 \left(\|D_A f\|^2 + (Vf, f) \right)$,

iii) $\forall f \in \mathcal{F}(X)$ *with* $\|D_A f\|^2 + (Vf, f) \leq \Lambda \|f\|^2$, $\|\mathcal{S}f\|^2 \geq c_2 \|f\|^2$ *holds.*

Proof: for the first inequality note that $\{B(p, \varepsilon)\}_{p \in X}$ covers M. Hence

$$\begin{aligned} \|\mathcal{S}f\|^2 &\leq \sum_{q \in X} \int_{B(q,\varepsilon)} \left| \sum_{p \in B(q,3\varepsilon) \cap X} \psi_p(x) \sum_{i=1}^{n} f_i(p) e_i^p(x) \right|^2 dV(x) \\ &\leq (1+\delta') \sum_{q \in X} V(q, \varepsilon) \sum_{p \in B(q,3\varepsilon) \cap X} |f(p)|^2 \leq (1+\delta') c \|f\|^2. \end{aligned}$$

In order to prove *ii)* fix $q \in X$ and let $x \in B(q, \varepsilon)$. Then as $\{\psi_p\}_{p \in X}$ is a partition of unity, we have $\sum_{p \in X} d\psi_p = 0$, so that we can write

$$\begin{aligned} \nabla(\mathcal{S}f)(x) &= \sum_{p \in B(q,3\varepsilon) \cap X} \psi_p(x) \Big(\sum_{i=1}^{n} f_i(p) \nabla e_i^p(x) \Big) + \\ & \sum_{p \in N(q)} d\psi_p(x) \Big(\sum_{i=1}^{n} f_i(p) e_i^p(x) - \sum_{i=1}^{n} f_i(q) e_i^q(x) \Big). \end{aligned} \tag{3.7}$$

Then, Lemma 3.12 implies

$$\int_{B(q,\varepsilon)} \left| \sum_{p\in B(q,3\varepsilon)\cap X} \psi_p(x) \Big(\sum_{i=1}^{n} f_i(p) \nabla e_i^p(x) \Big) \right|^2 dV(x) \leq$$

$$n \sum_{p\in B(q,3\varepsilon)\cap X} \left(\sum_{i\leq\mu(p)} f_i(p)^2 \int_{B(q,\varepsilon)} |\nabla e_i^p(x)|^2 dV(x) + c \sum_{i>\mu(p)} f_i(p)^2 \right)$$

$$\leq c' \sum_{p\in B(q,3\varepsilon)\cap X} (Vf)(p) \cdot f(p). \quad (3.8)$$

To estimate the second term of (3.7), we need the following lemma.

Lemma 3.19 *There exists a positive constant c depending only on m, n, k_1, k_2, κ and ε such that*

$$\int_{B(q,\varepsilon)} \left| \sum_{i=1}^{n} f_i(p) e_i^p(x) - \sum_{i=1}^{n} f_i(q) e_i^q(x) \right|^2 \leq$$

$$c \left(|D_A f(q,p)|^2 + (Vf)(p) \cdot f(p) + (Vf)(q) \cdot f(q) \right).$$

Proof: see Appendix A.1. □

Hence by (3.8), (3.7) and Lemma 3.19 we get

$$\int_{B(q,\varepsilon)} |\nabla(\mathcal{S}f)(x)|^2 dV(x) \leq$$

$$c'' \sum_{p\in B(q,3\varepsilon)\cap X} \Big(|D_A f(q,p)|^2 + (Vf)(p) \cdot f(p) + (Vf)(q) \cdot f(q) \Big).$$

Then summing on $q \in X$ implies the claim.

To prove the third part of Proposition 3.18, define $(\mathcal{S}_q f)(x) = \sum_{i=1}^{n} f_i(q) e_i^q(x)$ for x in $B(q, \frac{\varepsilon}{2})$. Then, by Lemma 3.19 we get

$$\int_{B(q,\frac{\varepsilon}{2})} |(\mathcal{S}f)(x) - (\mathcal{S}_q f)(x)|^2 \, dV(x) =$$

$$\int_{B(q,\frac{\varepsilon}{2})} \left| \sum_{p\in N(q)} \psi_p(x) \sum_{j=1}^{n} \left(f_j(p) e_j^p(x) - f_j(q) e_j^q(x) \right) \right|^2 dV(x) \leq$$

$$c \sum_{p\in N(q)} \left(|D_A f(q,p)|^2 + (Vf)(p) \cdot f(p) + (Vf)(q) \cdot f(q) \right). \quad (3.9)$$

As the balls of radius $\frac{\varepsilon}{2}$ centered on X are disjoint, we can write

$$\begin{aligned}\|\mathcal{S}f\|^2 &\geq \sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |(\mathcal{S}_q f(x) - \mathcal{S}f(x)) - \mathcal{S}_q f(x)|^2\, dV(x) \\ &\geq \sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |\mathcal{S}_q f(x)|^2\, dV(x) \\ &\quad -2\sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |\mathcal{S}_q f(x)|\,|\mathcal{S}f(x) - \mathcal{S}_q f(x)|\, dV(x).\end{aligned}$$

By construction, $(1-\delta')|f(q)|^2 \leq |\mathcal{S}_q f(x)|^2 \leq (1+\delta')|f(q)|^2$ and by Cauchy-Schwarz inequality combined with (3.9), we get

$$\sum_{q\in X}\int_{B(q,\frac{\varepsilon}{2})} |\mathcal{S}_q f(x)|\,|\mathcal{S}f(x) - \mathcal{S}_q f(x)|\, dV(x) \leq c'(1+\delta')\|f\|\sqrt{\|D_A f\|^2 + (Vf,f)}.$$

Hence, $\|\mathcal{S}f\|^2 \geq (1-\delta')c''\|f\|^2 - 2c'(1+\delta')\|f\|\sqrt{\|D_A f\|^2+(Vf,f)}$. Choose $\Lambda > 0$ sufficiently small so that if f satisfies $\|D_A f\|^2 + (Vf,f) \leq \Lambda\|f\|^2$, then

$$\|\mathcal{S}f\|^2 \geq \|f\|^2\left((1-\delta')c'' - 2c'(1+\delta')\sqrt{\Lambda}\right) \geq \frac{(1-\delta')c''}{2}\|f\|^2.$$

This concludes the proof of Proposition 3.18. □

3.4 Discretizing operator

Definition 3.20 *Define the* ***discretizing operator*** $\mathcal{D} : \Gamma(E) \to \mathcal{F}(X)$ *by*

$$(\mathcal{D}s)(p) = \sum_{i=1}^{n} \frac{1}{V(p,3\varepsilon)}\int_{B(p,3\varepsilon)} s_i^p(x)dV(x)e_i$$

where $s(x) = \sum_{i=1}^n s_i^p(x)e_i^p(x)$ *for* x *in* $B(p,3\varepsilon)$.

Proposition 3.21 *There exist constants* c_0', c_1', c_2' *and* $\Lambda' > 0$ *depending only on* m, n, κ, k_1, k_2 *and* ε *such that*

i) $\forall s \in \Gamma(E)$, $\|\mathcal{D}s\|^2 \leq c_0'\|s\|^2$,

ii) $\forall s \in \Gamma(E)$, $\|D_A(\mathcal{D}s)\|^2 + (V(\mathcal{D}s), \mathcal{D}s) \leq c_1'\|\nabla s\|^2$,

iii) $\forall s \in \Gamma(E)$ *such that* $\|\nabla s\|^2 \leq \Lambda'\|s\|^2$, $\|\mathcal{D}s\|^2 \geq c_2'\|s\|^2$ *holds.*

Proof: the first point follows directly from the following inequality

$$|\mathcal{D}s(p)|^2 \leq c \int_{B(p,3\varepsilon)} \sum_{i=1}^{n} |s_i^p(x)|^2 dV(x) \leq c(1-\delta')^{-1} \int_{B(p,3\varepsilon)} |s(x)|^2 dV(x).$$

To prove the second point, we first prove that

$$\|D_A(\mathcal{D}s)\|^2 + (V(\mathcal{D}s), \mathcal{D}s) \;\; \leq \;\; c \left(\|\nabla s\|^2 + \sum_{p \in X} \left(\widetilde{V} s \right)(p) \right) \tag{3.10}$$

where if **E is of harmonic curvature** then

$$\left(\widetilde{V} s \right)(p) = \left(\sum_{i \leq \mu(p)} \lambda_i(p) \int_{B(p,3\varepsilon)} |s_i^p|^2 dV + \sum_{i > \mu(p)} \int_{B(p,3\varepsilon)} |s_i^p|^2 dV \right)$$

and if **E is of complex (or quaternionic) rank one**

$$\left(\widetilde{V} s \right)(p) = \left(\lambda_1(p) + \sum_{q \in N(p)} \lambda_1(q) \right) \int_{B(p,3\varepsilon)} |s|^2 dV$$

and s is written locally as $s(x) = \sum_{i=1}^{n} s_i^p(x) e_i^p(x)$ for $x \in B(p, 8\varepsilon)$. First, $|(\mathcal{D}s)(p)_j|^2 \leq c \int_{B(p,3\varepsilon)} |s_j^p(x)|^2 dV(x)$ implies obviously

$$(V(\mathcal{D}s), \mathcal{D}s) \;\; \leq \;\; \sum_{p \in X} c' \left(\widetilde{V} s \right)(p). \tag{3.11}$$

Secondly, for p and $q \in N(p)$ let us introduce $B'_{pq} \subseteq B(p, 3\varepsilon) \cap B(q, 3\varepsilon)$ the ball centered at the mid-point of p and q of radius ε and V'_{pq} its volume. Then

$$|D_A(\mathcal{D}s)(q,p)|^2 = \sum_{i=1}^{n} \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| \mathcal{D}s(p)_i - \sum_{j=1}^{n} A(q,p)_{ij} \mathcal{D}s(q)_j \right| dV(y) \right)^2$$

$$\leq \;\; 3 \sum_{i=1}^{n} \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} |\mathcal{D}s(p)_i - s_i^p(y)| \, dV(y) \right)^2 \tag{3.12}$$

$$+ \;\; 3 \sum_{i=1}^{n} \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| \sum_{j=1}^{n} A(q,p)_{ij} \left(s_j^q(y) - \mathcal{D}s(q)_j \right) \right| dV(y) \right)^2 \tag{3.13}$$

$$+ \;\; 3 \sum_{i=1}^{n} \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^{n} A(q,p)_{ij} s_j^q(y) \right| dV(y) \right)^2 . \tag{3.14}$$

We estimate each of the three terms separately.

By a result of Kanai (see [10], Lemma VI.5.5), there exists $c_K > 0$ depending only on ε and κ such that

$$\frac{1}{V'_{pq}} \int_{B'_{pq}} |\mathcal{D}s(p)_i - s_i^p(y)|\, dV(y) \quad \leq \quad c_K \int_{B(p,3\varepsilon)} |ds_i^p(y)|\, dV(y).$$

Moreover

$$\sqrt{1-\delta'}\, |ds_i^p(y)| \leq \left| \sum_{j=1}^n ds_j^p(y) e_j^p(y) \right| = \left| \nabla s(y) - \sum_{j=1}^n s_j^p(y) \nabla e_j^p(y) \right|. \quad (3.15)$$

Therefore

$$\sqrt{1-\delta'} \int_{B(p,3\varepsilon)} |ds_i^p(y)|\, dV(y) \leq$$
$$\left(V(p,3\varepsilon) \int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) \right)^{\frac{1}{2}} + n \sum_{j=1}^n \|\nabla e_j^p\|_{2,3\varepsilon} \|s_j^p\|_{2,3\varepsilon}$$

so that we obtain by Lemma 3.12 and by construction of e_j^p

$$\sum_{i=1}^n \left(\int_{B(p,3\varepsilon)} |ds_i^p(y)|\, dV(y) \right)^2 \leq c \int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) + c\widetilde{V} s(p).$$

We have then the following upper bound for (3.12)

$$\sum_{i=1}^n \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} |\mathcal{D}s(p)_i - s_i^p(y)|\, dV(y) \right)^2$$
$$\leq c_K^2 c \left(\int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) + \left(\widetilde{V} s\right)(p) \right). \quad (3.16)$$

By the same kind of arguments as for (3.12) and using that $\sum_{i,j=1}^n |A(q,p)_{ij}|^2$ is bounded above by a uniform constant, we can bound (3.13) as follows

$$\sum_{i=1}^n \left(\frac{1}{V'_{pq}} \int_{B'_{pq}} \left| \sum_{j=1}^n A(q,p)_{ij} \left(s_j^q(y) - \mathcal{D}s(q)_j \right) \right| dV(y) \right)^2 \leq$$
$$c' \left(\int_{B(q,3\varepsilon)} |\nabla s(y)|^2 dV(y) + \left(\widetilde{V} s\right)(q) \right). \quad (3.17)$$

The last term (3.14) is then bounded by the following lemma

Lemma 3.22 *There exists a positive constant c depending only on m, n, k_1, k_2, κ and ε such that*

$$\sum_{i=1}^{n}\left(\int_{B'_{pq}}\left|s_i^p(y)-\sum_{j=1}^{n}A(q,p)_{ij}s_j^q(y)\right|dV(y)\right)^2 \leq c\left(\left(\widetilde{V}f\right)(p)+\left(\widetilde{V}f\right)(q)\right).$$

Proof: see Appendix A.2. □

Finally, (3.16), (3.17) and Lemma 3.22 imply that

$$\begin{aligned}|D_A(\mathcal{D}s)(p,q)|^2 \;&\leq\; c''\left(\int\limits_{B(p,3\varepsilon)}|\nabla s(y)|^2dV(y)+\int\limits_{B(q,3\varepsilon)}|\nabla s(y)|^2dV(y)\right)\\ &\quad + c''\left(\left(\widetilde{V}s\right)(p)+\left(\widetilde{V}s\right)(q)\right).\end{aligned}$$

Taking the sum over p and q leads to

$$\|D_A(\mathcal{D}s)\|^2 \;\leq\; c'''\left(\|\nabla s\|^2+\sum_{p\in X}\left(\widetilde{V}s\right)(p)\right) \tag{3.18}$$

so that (3.18),(3.11) imply (3.10). In order to conclude the proof of point $ii)$ of this lemma, we have to show that there exists $c>0$ such that

$$\sum_{p\in X}\left(\widetilde{V}s\right)(p)\leq c\|\nabla s\|^2. \tag{3.19}$$

Fix $q\in X$, let $B=B(q,10\varepsilon)$, $V(B)$ its volume. Let $(\cdot,\cdot)_B$ and $\|\cdot\|_B$ the L^2-inner product respectively the L^2-norm on E restricted to B. We are going to show that there exists $c>0$ such that

$$\left(\widetilde{V}s\right)(q)\leq c\sum_{p\in B(q,3\varepsilon)\cap X}\|\nabla s\|^2_{B(p,10\varepsilon)}. \tag{3.20}$$

Then (3.19) is a direct consequence of (3.20). To prove (3.20) we have to consider separately the cases E is of complex (or quaternionic) rank one and E is of harmonic curvature.

Assume E is of rank one. The proof of (3.20) in this case is much easier than in the other case as the potential involves only the first eigenvalue of

the ball. Recall that $\lambda_1(q) \leq \frac{\|\nabla s\|_B^2}{\|s\|_B^2}$ for any non-zero s. Therefore and as $B(q,3\varepsilon) \subseteq B(p,10\varepsilon)$ for any $p \in N(q)$

$$\left(\widetilde{V}s\right)(q) \;\leq\; \|s\|^2_{B(q,3\varepsilon)} \sum_{p\in B(q,3\varepsilon)\cap X} \frac{\|\nabla s\|^2_{B(p,10\varepsilon)}}{\|s\|^2_{B(p,10\varepsilon)}} \;\leq\; \sum_{p\in B(q,3\varepsilon)\cap X} \|\nabla s\|^2_{B(p,10\varepsilon)}$$

and this concludes the first case.

Assume E is of harmonic curvature. If $y \in B$, write $s(y)$ as a sum of orthogonal sections (with respect to $(\cdot,\cdot)_B$) $s(y) = \widetilde{s}(y) + s^\perp(y)$ with $\widetilde{s}(y) = \sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B}{V(B)} e_j^q(y)$. We have the following properties of the decomposition.

$$\begin{array}{rclrcl}
(s^\perp, e_j^q)_B &=& 0\ ,\ \ \forall j \leq \mu(q), & (\nabla s^\perp, \nabla\widetilde{s})_B &=& 0,\\
\|s\|_B^2 &=& \|s^\perp\|_B^2 + \|\widetilde{s}\|_B^2, & \|\nabla s\|_B^2 &=& \|\nabla s^\perp\|_B^2 + \|\nabla\widetilde{s}\|_B^2,\\
\|\widetilde{s}\|_B^2 &=& \displaystyle\sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)}, & \|\nabla\widetilde{s}\|_B^2 &=& \displaystyle\sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)}\lambda_j(q).
\end{array}$$

Consider then two cases. First assume $\|s^\perp\|_B^2 = 0$. Then $s(y) = \widetilde{s}(y)$ which means that if $y \in B(p,10\varepsilon)$

$$s_j^q(y) \;=\; \begin{cases} 0 & \text{if } j > \mu(q),\\ \frac{(s,e_j^q)_B}{V(B)} & \text{if } j \leq \mu(q).\end{cases}$$

Therefore

$$\begin{aligned}
\left(\widetilde{V}s\right)(q) \;&=\; \left(\sum_{j\leq\mu(q)} \lambda_j(q)\int_{B(q,3\varepsilon)} |s_j^q|^2 dV + \sum_{j>\mu(q)} \int_{B(q,3\varepsilon)} |s_j^q|^2 dV\right)\\
&=\; V(q,3\varepsilon)\sum_{j\leq\mu(q)} \frac{(s,e_j^q)_B^2}{V(B)^2}\lambda_j(q) \;\leq\; c\|\nabla\widetilde{s}\|_B^2.
\end{aligned}$$

Moreover as $s^\perp$ is zero $\|\nabla\widetilde{s}\|_B^2 = \|\nabla s\|_B^2$ and so in this case (3.20) is verified.

For the second case, assume $\|s^\perp\|_B^2 \neq 0$. Then apply max-min theorem to $s^\perp$ to obtain $\lambda_{\mu(q)+1}(q) \leq \frac{\|\nabla s^\perp\|_B^2}{\|s^\perp\|_B^2}$ and by definition of $\mu(q)$ this implies that

$$\delta\|s^\perp\|_B^2 \leq \|\nabla s^\perp\|_B^2. \tag{3.21}$$

Moreover, let us rewrite $s^\perp$ as follows, for $y \in B(q,8\varepsilon)$

$$s^\perp(y) \;=\; \sum_{j\leq\mu(q)} \left(s_j^q(y) - \frac{(s,e_j^q)_B}{V(B)}\right) e_j^q(y) + \sum_{j>\mu(q)} s_j^q(y)e_j^q(y).$$

As $\{e_j^q(y)\}$ is an almost orthonormal basis, we obtain for $y \in B(q, 8\varepsilon)$

$$\sum_{j \le \mu(q)} \left| s_j^q(y) - \frac{(s, e_j^q)_B}{V(B)} \right|^2 + \sum_{j > \mu(q)} |s_j^q(y)|^2 \;\; \le \;\; (1-\delta')^{-1} |s^\perp(y)|^2.$$

In particular, this implies

$$\sum_{j > \mu(q)} \int_{B(q,3\varepsilon)} |s_j^q(y)|^2 dV(y) \;\; \le \;\; (1-\delta')^{-1} \|s^\perp\|_B^2 \tag{3.22}$$

and

$$\begin{aligned} &\sum_{j \le \mu(q)} \lambda_j(q) \int_{B(q,3\varepsilon)} |s_j^q(y)|^2 dV(y) \le \\ &\quad 2 \sum_{j \le \mu(q)} \lambda_j(q) \int_{B(q,3\varepsilon)} \left| s_j^q(y) - \frac{(s, e_j^q)_B}{V(B)} \right|^2 dV(y) + 2 \sum_{j \le \mu(q)} \frac{(s, e_j^q)_B^2}{V(B)} \lambda_j(q) \\ &\qquad \le \frac{2\delta}{1-\delta} \|s^\perp\|_B^2 + 2 \|\nabla \widetilde{s}\|_B^2. \end{aligned} \tag{3.23}$$

Then (3.22) and (3.23) imply that $\left(\widetilde{V} s\right)(q) \;\le\; c \left(\|s^\perp\|_B^2 + \|\nabla \widetilde{s}\|_B^2\right)$. Use (3.21) together with this inequality to obtain (3.20) and therefore (3.19). Finally (3.10) together with (3.19) imply *ii*).

To prove *iii*) consider the following sum. By the work of Buser (Lemma 5.1 in [8]), there exists $c_B > 0$ depending only on m, κ and ε such that

$$\sum_{i=1}^n \int_{B(p,3\varepsilon)} |\mathcal{D}s(p)_i - s_i^p(x)|^2 \, dV(x) \;\; \le \;\; c_B \sum_{i=1}^n \int_{B(p,3\varepsilon)} |ds_i^p(x)|^2 \, dV(x).$$

Moreover, using (3.15) we obtain

$$\begin{aligned} &\sum_{i=1}^n \int_{B(p,3\varepsilon)} |\mathcal{D}s(p)_i - s_i^p(x)|^2 \, dV(x) \le \\ &\quad \frac{2nc_B}{1-\delta'} \left(\int_{B(p,3\varepsilon)} |\nabla s(y)|^2 dV(y) + n \sum_{j=1}^n \|\nabla e_j^p\|_{\infty,3\varepsilon}^2 \|s_j^p(y)\|_{2,3\varepsilon}^2 \right). \end{aligned} \tag{3.24}$$

Therefore, from (3.24) we obtain

$$|\mathcal{D}s(p)|^2 \geq c \int\limits_{B(p,3\varepsilon)} \sum_{i=1}^{n} |(s_i^p(x) - \mathcal{D}s(p)_i) - s_i^p(x)|^2 \, dV(x)$$

$$\geq c \int\limits_{B(p,3\varepsilon)} \sum_{i=1}^{n} |s_i^p(x)|^2 dV(x) - 2c \int\limits_{B(p,3\varepsilon)} \sum_{i=1}^{n} |s_i^p(x)||\mathcal{D}s(p)_i - s_i^p(x)| dV(x)$$

$$\geq c' \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \left(\|\nabla s\|^2_{B(p,3\varepsilon)} + \sum_{j=1}^{n} \|\nabla e_j^p\|^2_{\infty,3\varepsilon} \|s_j^p\|^2_{2,3\varepsilon} \right)^{\frac{1}{2}} \quad (3.25)$$

Assume E is of harmonic curvature and combine Lemma 3.3 and Lemma 3.12 with (3.25) to obtain

$$|\mathcal{D}s(p)|^2 \geq c' \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \left(\|\nabla s\|^2_{B(p,3\varepsilon)} + \left(\widetilde{V} s\right)(p) \right)^{\frac{1}{2}}.$$

Moreover, by (3.20) $\left(\widetilde{V} s\right)(p)$ is bounded above by $c \sum\limits_{q \in B(p,3\varepsilon)\cap X} \|\nabla s\|^2_{B(q,10\varepsilon)}$. Then, taking the sum over $p \in X$ produces new c', $c'' > 0$ such that

$$\|\mathcal{D}s\|^2 \;\; \geq \;\; c'\|s\|^2 - c''\|s\|\|\nabla s\|.$$

Finally, if $\|\nabla s\|^2 \leq \Lambda' \|s\|^2$, we get $\|\mathcal{D}s\|^2 \geq \|s\|^2 (c' - c''\sqrt{\Lambda'})$. Choose then Λ' suitably to conclude the proof of the proposition in this case.

Assume E is of rank one. If $\lambda_1(p) \leq \delta$, by Lemma 3.3, $\|\nabla e_j^p\|^2_{\infty,3\varepsilon} \leq c\lambda_1^s(p)$. If $\lambda_1(p) > \delta$, by Lemma 3.12 $\|\nabla e_j^p\|^2_{\infty,3\varepsilon} \leq c \leq c\delta^{-1}\lambda_1(p)$. Therefore, (3.25) can be changed in (with new constants c, c', c'')

$$|\mathcal{D}s(p)|^2 \geq \begin{cases} (c' - c\lambda_1^{\frac{s}{2}}(p)) \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \|\nabla s\|_{B(p,3\varepsilon)} & \text{if } \lambda_1(p) \leq \delta, \\ \\ c' \|s\|^2_{B(p,3\varepsilon)} - c'' \|s\|_{B(p,3\varepsilon)} \|\nabla s\|_{B(p,10\varepsilon)} & \text{otherwise.} \end{cases}$$

By choosing δ smaller, we can assume that if $\lambda_1(p) \leq \delta$, $c' - c\lambda_1(p)^{\frac{s}{2}} \geq c''' > 0$. This implies that (for any values of $\lambda_1(p)$)

$$|\mathcal{D}s(p)|^2 \;\; \geq \;\; c''' \|s\|^2_{B(p,3\varepsilon)} - c'' \|\nabla s\|_{B(p,3\varepsilon)} \|s\|_{B(p,10\varepsilon)}.$$

Then, take the sum over $p \in X$ to obtain for $\|\nabla s\| \leq \Lambda' \|s\|$

$$\|\mathcal{D}s\|^2 \;\; \geq \;\; c''' \|s\|^2 - c'' \|\nabla s\| \|s\| \;\; \geq \;\; \|s\|^2 (c''' - c''\sqrt{\Lambda'})$$

and conclude choosing Λ' suitably. $\square$

3.5 Upper bounds

Lemma 3.23 *Let m, n, k_1, k_2, κ, r_0, ε be as before. Then there exist positive constants c_3 and c_3' depending only on m, n, k_1, k_2, κ, ε so that for any vector bundle $E \in \mathcal{E}(n, k_1, k_2)$ over any $M \in \mathcal{M}(m, \kappa, r_0)$, for any X ε-discretization of E and for $\Delta_A + V$ constructed in Section 3.2, we have*

i) $\lambda_k(E) \le c_3$, $\forall k \le n|X|$,

ii) $\lambda_k(X, A, V) \le c_3'$, $\forall k \le n|X|$.

Proof: *i)* Let p_i be a vertex of X and consider $f_i : M \to \mathbb{R}$ the first eigenfunction of the Dirichlet problem for the ball centered at p_i of radius $\frac{\varepsilon}{2}$ extended by zero. By Cheng's comparison theorem $\frac{\|df_i\|^2}{\|f_i\|^2} \le \lambda_1\left(\frac{\varepsilon}{2}, \kappa\right)$ (where $\lambda_1\left(\frac{\varepsilon}{2}, \kappa\right)$ denotes the first non-zero eigenvalue of the Dirichlet problem on the ball of radius $\frac{\varepsilon}{2}$ in the simply connected space of constant sectional curvature $-\kappa$ and of same dimension as M). Define then the sections $\sigma_j^i(x) = f_i(x)e_j^{p_i}(x)$ for $1 \le i \le |X|$, and $1 \le j \le n$. Then $\{\sigma_j^i \mid 1 \le i \le |X|, 1 \le j \le n\}$ spans a vector subset W of $\Gamma(E)$ of dimension $n|X|$ as $\{e_j^{p_i}\}_{j=1,\dots,n}$ is an almost orthonormal frame. Moreover

$$\nabla\sigma_j^i(x) \;=\; df_i(x)e_j^{p_i}(x) + f_i(x)\nabla e_j^{p_i}(x)$$

hence by construction of $e_j^{p_i}$ and Lemma 3.3 and Lemma 3.12, we have

$$\|\nabla\sigma_j^i\|^2 \;\le\; c\left(\|df_i\|^2 + \|f_i\|^2\right)$$

so that by definition of the f_i's

$$\|\nabla\sigma_j^i\|^2 \;\le\; c\|f_i\|^2\left(1 + \lambda_1\left(\frac{\varepsilon}{2}, \kappa\right)\right).$$

By min-max theorem we get then

$$\lambda_k(E) \;\le c' \max\left\{\frac{\sum_{i,j} a_{ij}^2\|\nabla\sigma_j^i\|^2}{\sum_{i,j} a_{ij}^2\|\sigma_j^i\|^2}\right\} \;\le c'c\left(1 + \lambda_1\left(\frac{\varepsilon}{2}, \kappa\right)\right).$$

This concludes the first part of the lemma.

ii) Let $f \in \mathcal{F}(X)$. As $A(p,q)$ is a change of almost orthonormal bases we have

$$\begin{aligned}
\|D_A f\|^2 + (Vf, f) &= \frac{1}{2}\sum_{p\in X}\sum_{q\in N(p)} |f(q) - A(p,q)f(p)|^2 + \sum_{p\in X}(Vf)(p)\cdot f(p) \\
&\le c\sum_{p\in X}\sum_{q\in N(p)}\left(|f(p)|^2 + |f(q)|^2\right) + \max\{\delta, 1\}\|f\|^2 \\
&\le \left(2c\nu_X + \max\{\delta, 1\}\right)\|f\|^2.
\end{aligned}$$

Therefore, $R(f) \le 2c\nu_X + \max\{\delta, 1\}$, $\forall f \in \mathcal{F}(X) \setminus \{0\}$ and this implies $\lambda_k(X, A, V) \le 2c\nu_X + \max\{\delta, 1\}$, $\forall k \le n|X|$. $\square$

3.6 Conclusion

Proof of Theorem 3.1: by symmetry of the results concerning the smoothing and the discretizing, it suffices to deduce $\lambda_k(E) \leq c\lambda_k(X, A, V)$. The proof proceeds in two steps.

First, assume that k is such that $\lambda_k(X, A, V) \geq \Lambda$, for Λ given by Proposition 3.18 iii). Then, Lemma 3.23 i) leads to $\lambda_k(E) \leq c_3\Lambda^{-1}\lambda_k(X, A, V)$.This is the required inequality.

Secondly, assume that k is such that $\lambda_k(X, A, V) \leq \Lambda$. Let W_k be the k-dimensional vector subspace of $\mathcal{F}(X)$ spanned by $f_i : X \to \mathbb{R}^n$, $i = 1, \ldots, k$, $\lambda_i(X, A, V)$-eigenfunction of Δ_A chosen so that $(f_i, f_j) = \delta_{ij}|X|$. By min-max theorem, $\lambda_k(X, A, V) = \max\{R(f) : f \in W_k \setminus \{0\}\}$. Let then $\mathcal{S}W_k$ be the vector subspace of $\Gamma(E)$ spanned by the $\mathcal{S}f_i$'s i.e. $\mathcal{S}W_k = \langle \mathcal{S}f_1, \ldots, \mathcal{S}f_k\rangle = \{\mathcal{S}f \mid f \in W_k \setminus \{0\}\}$. As $\lambda_k(X, A, V) \leq \Lambda$, for any non-zero function f in W_k, we have $\|D_A f\|^2 + (Vf, f) \leq \Lambda\|f\|^2$. Hence, by Proposition 3.18 iii), for any f in W_k, $\|\mathcal{S}f\|^2 \geq c_2\|f\|^2$ holds. In particular, $\mathcal{S}f$ is the zero function if and only if f is zero which means that $\mathcal{S}W_k$ is k-dimensional. So we can apply min-max theorem to $\mathcal{S}W_k$ and obtain

$$\lambda_k(E) \leq \max\{R(\mathcal{S}f) \mid f \in W_k \setminus \{0\}\}.$$

Moreover, by Proposition 3.18 ii) and iii) we obtain that $R(\mathcal{S}f) \leq \frac{c_1}{c_2}R(f)$ for any non-zero f in W_k, which leads to

$$\lambda_k(E) \leq \frac{c_1}{c_2}\max\{R(f) \mid f \in W_k \setminus \{0\}\} = \frac{c_1}{c_2}\lambda_k(X, A, V).$$

This concludes the proof. □

4 Estimation of the first non-zero eigenvalue for a flat vector bundle

Let (E^n, ∇) be a flat Riemannian vector bundle with irreducible holonomy over $M \in \mathcal{M}(m, \kappa, r_0)$. We recall the definition of the constant related to the holonomy given by Ballmann, Brüning and Carron in [2]. If c is a unit speed loop, denote by H_c its holonomy. Then there exists $\alpha > 0$ such that $\forall x \in M$, $\forall v \in E_x$ there exists a smooth unit speed loop $c_{x,v}$ of length less than two diameters of M such that

$$|H_{c_{x,v}}(v) - v| \geq \alpha|v|. \tag{4.1}$$

The following theorem shows that if E has significant holonomy, then the first eigenvalue of $\overline{\Delta}$ can not be too small. Conversely, if there exists v in E_x which has a small holonomy, then the first eigenvalue is not too large.

Theorem 4.1 *Let (E^n, ∇) be a flat Riemannian vector bundle over $M \in \mathcal{M}(m, \kappa, r_0)$ with irreducible holonomy. Then there exist c, $c' > 0$ depending only on m, n, κ, r_0 such that*

$$\lambda_1(E) \geq c' \frac{\alpha^2}{d(M)^2 c^{d(M)}}$$

where $d(M)$ denotes the diameter of M.

Moreover, if there exist $p_0 \in M$, $v_0 \in E_{p_0}$ and α' such that for any loop c at p_0 of length less than $7d(M)$, $|H_c(v_0) - v_0| \leq \alpha' |v_0|$ then, there exists $c'' > 0$ depending only on n, m, κ and r_0 such that

$$\lambda_1(E) \leq c'' \alpha'^2.$$

The first part of the theorem is in fact due to Ballmann, Brüning and Carron (see [2]). We present here a more conceptual proof that relies on the fact that the discrete magnetic Laplacian associated to a discretization of a flat bundle is strongly related to the holonomy of the vector bundle.

Proof: let $\varepsilon = \frac{1}{100} r_0$ and let X be an ε-discretization of E. Then by Theorem 3.1 there exist Δ_A a discrete magnetic Laplacian and $c > 0$ such that $\lambda_1(E) \geq c\lambda_1(X, A)$. So it suffices to prove the statement for $\lambda_1(X, A)$. Let $f \in \mathcal{F}(X)$ such that $\Delta_A f = \lambda f$. Let $p_0 \in X$ and $v_0 = \sum_{i=1}^n f_i(p_0) e_i^{p_0} \in E_{p_0}$. By (4.1), there exists a smooth unit speed loop $c_0 : [0, l] \to M$ at p_0 of length $l \leq 2d(M)$ and $|H_{c_0}(v_0) - v_0| \geq \alpha |v_0|$. Let $N \in \mathbb{N}$ such that $N\frac{\varepsilon}{2} \leq l < (N+1)\frac{\varepsilon}{2}$ and consider a partition of $[0, l]$, $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = l$ such that $\frac{\varepsilon}{2} \leq t_j - t_{j-1} \leq \varepsilon$. By definition of X, $\forall j = 1, \ldots, N-1$, $\exists p_j \in X$ such that $d(p_j, c_0(t_j)) < \varepsilon$. Moreover, let $p_N = p_0 \in X$. Note that $d(p_{j-1}, p_j) < 3\varepsilon$. Consider then the piecewise geodesic loop $\overline{c}_0$ at p_0 passing through all p_j, $j = 1, \ldots N-1$ (i.e $\overline{c}_0$ joins p_{j-1} to p_j via the minimizing geodesic $p_{j-1}p_j$). Note that $\overline{c}_0$ is of length less than $3N\varepsilon \leq 12d(M)$. Moreover, as E is flat, the holonomy of c_0 is the same as the holonomy of $\overline{c}_0$. More precisely, parallel translation from $c_0(t_{j-1})$ to $c_0(t_j)$ along c_0 is the same as parallel translation along minimizing geodesics from $c_0(t_{j-1})$ to p_{j-1}, then from p_{j-1} to p_j and finally from p_j to $c_0(t_j)$. Hence $H_{c_0}(v) = H_{\overline{c}_0}(v)$ for any $v \in E_{p_0}$. So that we obtain

$$|H_{\overline{c}_0}(v_0) - v_0| \geq \alpha |v_0| = \alpha |f(p_0)|.$$

Consider then $v_j = \sum_{i=1}^n f_i(p_j) e_i^{p_j} \in E_{p_j}$. By triangle inequality and as parallel transport is an isometry, we obtain easily the following inequality

$$\alpha |f(p_0)| \leq \sum_{j=1}^N |\tau_{p_j,p_{j-1}} v_{j-1} - v_j|.$$

Moreover, by construction of D_A we have

$$\begin{aligned}
|\tau_{p_j,p_{j-1}} v_{j-1} - v_j| &= \left| \sum_{i=1}^n f_i(p_{j-1}) \tau_{p_j,p_{j-1}} e_i^{p_{j-1}} - \sum_{i=1}^n f_i(p_j) e_i^{p_j} \right| \\
&= \left| \sum_{i=1}^n \left(\sum_{k=1}^n A(p_{j-1}, p_j)_{ik} f_k(p_{j-1}) - f_i(p_j) \right) e_i^{p_j} \right| \\
&= |D_A f(p_{j-1}, p_j)|.
\end{aligned}$$

This implies that $\alpha |f(p_0)| \leq |D_A f(p_0, p_1)| + \ldots + |D_A f(p_{N-1}, p_N)|$. We have shown that for any $p_0 \in X$, there exists a piecewise geodesic loop $\overline{c}_0 = \{p_0, p_1, \ldots, p_N\}$ of length less than $12d(M)$ such that

$$\alpha^2 |f(p_0)|^2 \leq 4 \frac{d(M)}{\varepsilon} \left(|D_A f(p_0, p_1)|^2 + \ldots + |D_A f(p_{N-1}, p_N)|^2 \right)$$

and $d(p_{j-1}, p_j) < 3\varepsilon$. The goal is to apply this last inequality to $\|f\|^2$. To that end, we need to find an upper bound for the number of loops of the kind $\{p, q, \ldots, p\}$ that can pass through $p \in X$ and $q \in N(p)$ and of length less than $12d(M)$. This upper bound on the length of the loop implies that such a loop can pass through at most $P \leq 12 \frac{d(M)}{\varepsilon}$ points of X. Therefore, there are at most ν^{P-1} loops of the kind $\{p, q, \ldots, p\}$ and each of these loops is suitable for P points in X. Hence, we obtain

$$\begin{aligned}
\alpha^2 \|f\|^2 &\leq P \nu^{P-1} 8 \frac{d(M)}{\varepsilon} \|D_A f\|^2 \\
&\leq 72 \frac{d(M)^2}{\varepsilon^2} \nu^{12 \frac{d(M)}{\varepsilon}} \|D_A f\|^2.
\end{aligned}$$

This leads then to the conclusion of the first part $\alpha^2 \frac{\varepsilon^2}{72 d(M)^2 \nu^{12 \frac{d(M)}{\varepsilon}}} \leq \lambda$.

To prove the second part of the theorem let $\varepsilon = \frac{1}{100} r_0$ and X be an ε-discretization of E such that $p_0 \in X$. Recall that X is the set of vertices of a finite connected graph G. Then construct a spanning tree S of G (see [4], Section I.2) as follows. Let $X_i = \{p \in X \mid d_G(p, p_0) = i\}$ where d_G denotes the path metric on G. Note that if q is in X_i then there exists q' in

X_{i-1} which is joined by an edge to q. Let then S be the subgraph of G with vertices set X and edges set $E(S) = \{qq' \mid q \neq p_0\}$. We have constructed a spanning tree S of G.

By construction of S, for any p in X there exists a unique curve c_p in S joining p to p_0 (i.e. c_p is a piecewise geodesic curve $\{p, \ldots, p_0\}$ such that two consecutive points of X in c_p are joined in S). Moreover the length of such a c_p is bounded above by $3d(M)$. Now, choose in E_{p_0} an orthonormal basis $\{e_1^{p_0}, \ldots, e_n^{p_0}\}$ and define an orthonormal basis $\mathcal{B}_p$ of E_p by $\mathcal{B}_p = \{e_i^p = \tau_{c_p} e_i^{p_0}\}_{i=1,\ldots,n}$, where τ_{c_p} denotes parallel transport along c_p from p_0 to p. Then $e_i^p(x) = \tau_{x,p} e_i^p$ gives a local orthonormal frame made of parallel sections. Hence, consider the discrete magnetic Laplacian Δ_A associated to this choice of bases (constructed as in Section 3.2) which satisfies $\lambda_1(E) \leq c\lambda_1(X, A)$ by Theorem 3.1. So that it suffices to prove the result for the first eigenvalue of Δ_A. By min-max theorem $\lambda_1(X, A) \leq R(f)$ for any non-zero function on X. So consider $f : X \to \mathbb{R}^n$ defined by $f(p) = \sum_{i=1}^n v_i e_i$ where the v_i's are the coordinates of v_0 in the basis $\mathcal{B}_{p_0}$. If p and q are neighboring points in X such that $d(p, p_0) \leq d(q, p_0)$ and $p \in c_q$, then we have $\tau_{q,p} e_j^p = e_j^q$. Hence in this case $A(p, q)_{ij} = \delta_{ij}$ and so $D_A f(p, q) = 0$. In the other case i.e. if $p \in N(q)$, $d(p, p_0) \leq d(q, p_0)$ and p is not on c_q, consider the loop c at x_0 going from x_0 to p via c_p, from p to q via the minimizing geodesic pq and from q to x_0 via c_q^{-1}. Then c is of length less than $7d(M)$ and by assumption

$$|H_c(v_0) - v_0| \leq \alpha' |v_0|. \tag{4.2}$$

But, we have $H_c(v_0) = \tau_{c_q}^{-1} \tau_{q,p} \tau_{c_p} v_0$ and

$$\langle H_c(v_0), e_i^{p_0} \rangle = \left\langle \sum_{j=1}^n \tau_{q,p} e_j^p, e_i^q \right\rangle = \sum_{j=1}^n A(p, q)_{ij} v_j.$$

Combining this last equality with (4.2) we obtain $\alpha' |v_0| \geq |D_A f(p, q)|$. Finally, computing $\|D_A f\|^2$ leads to

$$\|D_A f\|^2 \leq \frac{1}{2} \alpha'^2 \nu \|f\|^2.$$

So that the second part of the theorem follows. □

A Appendix: technical tools

The following lemma is a generalization of Lemma 11.1 in [20] and a local version of Lemma 0.1 of [27].

Lemma A.1 *Let $M \in \mathcal{M}(m, \kappa, r_0)$ and u a non-negative function on the ball $B(p, R)$, with $R < \frac{1}{2} r_0$, such that $\Delta u \leq \alpha u + \beta$. Let $0 < \theta < 1$. Then there exist c_1, c_2, $c_3 > 0$ (depending only on m, n, κ, R, α and β) and $0 < c(m) < s \leq 1$ such that*

$$\|u\|_{\infty,\theta R} \leq \left(\left(c_1 + c_2 \frac{1}{(1-\theta)^2} \right)^{c_3} \|u\|_{2,R} \right)^s$$

where $\|u\|_{\infty,\theta R} = \sup\{u(x) \mid x \in B(p, \theta R)\}$, and $\|u\|_{q,R}^q = \int_{B(p,R)} u^q(x) dV(x)$.

Note that, if $\beta = 0$ then $s = 1$ (see [20], Lemma 11.1).

Proof: the proof combines the proof given in [20] (Lemma 11.1) and Lemma 0.1 of [27] . Let $u : B(p, R) \to \mathbb{R}$, $u \geq 0$ such that $\Delta u \leq \alpha u + \beta$. Let $\nu = \frac{m}{2}$ if $m \geq 3$ and $\nu = 2$ otherwise. Let μ be such that $\frac{1}{\mu} + \frac{1}{\nu} = 1$. For $0 < \rho < \rho + \sigma < R$, let $\phi_{\rho,\sigma}$ be the Lipschitz cut-off function depending only on the distance to p given by

$$\phi_{\rho,\sigma}(r) = \phi(r) = \begin{cases} 0 & \text{on } B(p, R) \setminus B(p, \rho + \sigma), \\ \frac{\rho+\sigma+r}{\sigma} & \text{on } B(p, \rho + \sigma) \setminus B(p, \rho), \\ 1 & \text{on } B(p, \rho). \end{cases}$$

Then for an arbitrary constant $a \geq 1$, we have

$$\|u^{2a}\|_{\mu,\rho} \leq \|\phi u^a\|_{2\mu}^2.$$

As the injectivity radius of M is bounded below ($Inj(M) \geq r_0 > 0$) and the Ricci curvature too ($Ricci(M, g) \geq -(m-1)\kappa g$) Sobolev embeddings for complete manifolds are valid and we can apply the Sobolev inequalities to $\|\phi u^a\|_{2\mu}^2$ (see [16], lemma 3.3). More precisely, there exists a constant $c_s > 0$ depending only on m, κ and r_0 such that

$$\|\phi u^a\|_{2\mu}^2 \leq c_s \|d(\phi u^a)\|_2^2.$$

Replacing c_s by CR^2, we can rewrite the inequality as

$$\|\phi u^a\|_{2\mu}^2 \leq CR^2 \|d(\phi u^a)\|_2^2.$$

Therefore,

$$\|u^{2a}\|_{\mu,\rho} \leq CR^2 \|d(\phi u^a)\|_2^2.$$

However

$$\int_M |d(\phi u^a)|^2 dV \leq a \int_M \phi^2 u^{2a-1} \Delta u dV + \int_M |d\phi|^2 u^{2a} dV$$

(see [20], p.81). Hence using the assumption on Δu and $u \geq 0$ we obtain

$$\|u^{2a}\|_{\mu,\rho} \leq CR^2 \left(a\alpha \int_M \phi^2 u^{2a} dV + a\beta \int_M \phi^2 u^{2a-1} dV + \int_M |d\phi|^2 u^{2a} dV \right)$$

and by construction of ϕ, we obtain

$$\begin{aligned}\|u^{2a}\|_{\mu,\rho} &\leq CR^2 \left(a\alpha + \frac{1}{\sigma^2} \right) \int_{B(p,\rho+\sigma)} u^{2a} dV + CR^2 a\beta \int_{B(p,\rho+\sigma)} u^{2a-1} dV \\ &\leq CR^2 \left(a\alpha + \frac{1}{\sigma^2} \right) \|u\|_{2a,\rho+\sigma}^{2a} + CR^2 a\beta V(p,\rho+\sigma)^{\frac{1}{2a}} \|u\|_{2a,\rho+\sigma}^{2a-1}.\end{aligned}$$

Finally, we have shown that for any $a \geq 1$, $0 < \rho < \rho + \sigma < R$, we have

$$\|u\|_{2a\mu,\rho}^{2a} \;\; \leq \;\; CR^2 \left(a\alpha + \frac{1}{\sigma^2} \right) \|u\|_{2a,\rho+\sigma}^{2a} + CR^2 a\beta V(p,\rho+\sigma)^{\frac{1}{2a}} \|u\|_{2a,\rho+\sigma}^{2a-1}.$$

This was the first step of the proof. Now, we will proceed with a Moser iteration. To that aim, let

$$\begin{aligned} a_0 = 1,\ a_1 = \frac{m}{m-2} = \mu, \quad \ldots, \quad a_i = \mu^i, \ldots \\ \sigma_0 = \frac{1-\theta}{2} R,\ \sigma_1 = \frac{1-\theta}{4} R, \quad \ldots, \quad \sigma_i = \frac{1-\theta}{2^{i+1}} R, \ldots \\ \rho_0 = R - \sigma_0,\ \rho_1 = R - \sigma_0 - \sigma_1, \quad \ldots, \quad \rho_i = R - \sum_{j=0}^{i} \sigma_j, \ldots \end{aligned}$$

and $\rho_{-1} = R$. Observe that $\rho_i > \theta R$ for any i and $\rho_i \to \theta R$ as $i \to \infty$. Moreover, for any A_i, $B_i > 0$

$$\begin{aligned}(A_i + B_i) \min\{\|u\|_{2a_i,\rho_i+\sigma_i}^{2a_i}, \|u\|_{2a_i,\rho_i+\sigma_i}^{2a_{i-1}}\} \leq \\ A_i \|u\|_{2a_i,\rho_i+\sigma_i}^{2a_i} + B_i \|u\|_{2a_i,\rho_i+\sigma_i}^{2a_{i-1}} \leq (A_i + B_i) \|u\|_{2a_i,\rho_i+\sigma_i}^{b_i}\end{aligned}$$

where b_i is suitably chosen ($b_i \in \{2a_{i-1}, 2a_i\}$). Now replace above a, ρ, σ by a_i respectively ρ_i, σ_i to obtain

$$\|u\|_{2a_{i+1},\rho_i} \leq \left(CR^2 \left(a_i\alpha + \frac{1}{\sigma_i^2} + a_i\beta V(p,\rho_{i-1})^{\frac{1}{2a_i}} \right) \right)^{\frac{1}{2a_i}} \|u\|_{2a_i,\rho_{i-1}}^{\frac{b_i}{2a_i}}.$$

Then iterate this inequality to obtain (using Bishop-Gromov comparison theorem, Croke's inequality and $a_i \geq 1$)

$$\|u\|_{\infty,\theta R} \leq c \left(\prod_{i=0}^{\infty} \left(CR^2 a_i (\alpha + c'\beta) + C\frac{R^2}{\sigma_i^2} \right)^{\frac{1}{2a_i}} \|u\|_{2,R}^{\frac{b_0}{2}} \right)^{\prod_{j=1}^{\infty} \frac{b_j}{2a_j}}.$$

By the same argument as in [27], $\prod_{j=0}^{\infty} \frac{b_j}{2a_j}$ converges to $s \in [e^{-(n-2)\frac{\ln(2)}{2}}, 1]$. It remains then to show that $\prod_{i=0}^{\infty} \left(CR^2 a_i \left(\alpha + c'\beta\right) + C\frac{R^2}{\sigma_i^2} \right)^{\frac{1}{2a_i}}$ converges too. But we have that $\prod_{i=0}^{\infty} B^{\mu^{-i}} = B^{\frac{\mu}{\mu-1}}$ (as $\mu > 1$) and $\sum_{i=0}^{\infty} i\mu^{-i}$ is finite, therefore

$$
\begin{aligned}
\prod_{i=0}^{\infty} \left(CR^2 \mu^i \left(\alpha + c'\beta\right) + 4C\frac{4^i}{(1-\theta)^2} \right)^{\frac{1}{2\mu^i}} &\leq \\
\prod_{i=0}^{\infty} \max\{\mu, 4\}^{\frac{i}{2\mu^i}} \left(CR^2 \left(\alpha + c'\beta\right) + C\frac{4}{(1-\theta)^2} \right)^{\frac{1}{2\mu^i}} \\
\leq c(\mu) \left(CR^2 \left(\alpha + c'\beta\right) + C\frac{4}{(1-\theta)^2} \right)^{\frac{1}{2}\frac{\mu}{\mu-1}}.
\end{aligned}
$$

This implies the claim. □

A.1 Proof of Lemma 3.19

The proof differs according to the assumptions made on E.

Assume E is of harmonic curvature. By Remark 3.13 and Remark 3.14, we have

$$
\begin{aligned}
&\sum_{i=1}^{n} f_i(p) e_i^p(x) - \sum_{i=1}^{n} f_i(q) e_i^q(x) = \\
&\sum_{i=1}^{n} f_i(p) \left(e_i^p(x) - \tau_{x,p} e_i^p(p)\right) + D_A f(q,p)_i \tau_{x,p} e_i^p(p) + f_i(q) \left(\tau_{x,p} e_i^q(p) - e_i^q(x)\right).
\end{aligned}
$$

By Lemma 3.3 and as $d^* R^E = 0$, $|e_i^p(x) - \tau_{x,p} e_i^p|^2 \leq c\lambda_i(p)$ for $1 \leq i \leq \mu(p)$ and $|\tau_{x,p} e_i^q(p) - e_i^q(x)|^2 \leq c\lambda_i(q)$ for $1 \leq i \leq \mu(q)$. Moreover if $\mu(q) < i \leq n$, $|\tau_{x,p} e_i^q(p) - e_i^q(x)|^2 \leq 4$. Therefore

$$
\begin{aligned}
&\left| \sum_{i=1}^{n} f_i(p) e_i^p(x) - \sum_{i=1}^{n} f_i(q) e_i^q(x) \right|^2 \leq \\
&\qquad c' \left(\left| D_A f(q,p) \right|^2 + \left(V f\right)(p) \cdot f(p) + \left(V f\right)(q) \cdot f(q) \right)
\end{aligned}
$$

which implies the lemma in this case.

Assume E is of rank one, then

$$\sum_{i=1}^{n} f_i(p)e_i^p(x) - \sum_{i=1}^{n} f_i(q)e_i^q(x) = \\ \sum_{i=1}^{n} D_A f(q,p)_i e_i^p(x) + \sum_{j=1}^{n} f_j(q) \sum_{i=1}^{n} e_i^p(x) \left(A(q,p)_{ij} - a(q,p)_{ij}(x) \right).$$

By definition of $A(q,p)_{ij}$ and by the work of Buser (Lemma 5.1 in [8]) there exists $c_B > 0$ depending only on m, κ and ε such that

$$\int_{B_{pq}} \left| A(q,p)_{ij} - a(q,p)_{ij}(x) \right|^2 dV(x) \leq c_B \int_{B_{pq}} \left| da(q,p)_{ij}(x) \right|^2 dV(x).$$

Moreover

$$(1-\delta') \sum_{i=1}^{n} \left| da(q,p)_{ij}(x) \right|^2 \leq \\ \left| \sum_{i=1}^{n} da(q,p)_{ij}(x) e_i^p(x) \right|^2 = \left| \nabla e_j^q(x) - \sum_{i=1}^{n} a(q,p)_{ij}(x) \nabla e_i^p(x) \right|^2 \\ \leq c \left(\left| \nabla e_j^q(x) \right|^2 + \sum_{i=1}^{n} \left| \nabla e_i^p(x) \right|^2 \right).$$

As the bundle is of rank one, $\lambda_1(p) = \ldots = \lambda_n(p)$. Therefore $\lambda_1(p) \leq \delta$ implies $\int\limits_{B(p,10\varepsilon)} \left| \nabla e_i^p(x) \right|^2 dV(x) \leq c\lambda_1(p)$. Otherwise $\int\limits_{B(p,10\varepsilon)} \left| \nabla e_i^p(x) \right|^2 dV(x) \leq c \leq c\delta^{-1}\lambda_1(p)$ by Lemma 3.12, which implies

$$\int_{B_{pq}} \left| A(q,p)_{ij} - a(q,p)_{ij}(x) \right|^2 dV(x) \leq c' \left(\lambda_1(p) + \lambda_1(q) \right). \tag{A.1}$$

Hence

$$\int_{B(q,\varepsilon)} \left| \sum_{i=1}^{n} f_i(p)e_i^p(x) - \sum_{i=1}^{n} f_i(q)e_i^q(x) \right|^2 \leq \\ c'' \left(\left| D_A f(q,p) \right|^2 + \left| f(q) \right|^2 \left(\lambda_1(p) + \lambda_1(q) \right) \right).$$

This concludes the proof of Lemma 3.19. □

A.2 Proof of Lemma 3.22

The proof differs according to the assumptions made on E.

Assume E is of harmonic curvature. As $\{\tau_{y,p}e_i^p(p)\}_{i=1}^n$ is an almost orthonormal basis and by Remark 3.14

$$\sum_{i=1}^n \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right|^2 \leq$$

$$(1-\delta')^{-1} \left| \sum_{i=1}^n s_i^p(y) \left(\tau_{y,p}e_i^p(p) - e_i^p(y)\right) + \sum_{i=1}^n s_i^q(y) \left(e_i^q(y) - \tau_{y,p}e_i^q(p)\right) \right|^2 .$$

Integrate then over B'_{pq} and apply Lemma 3.3 to obtain

$$\sum_{i=1}^n \int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right|^2 dV(y) \leq c \left(\left(\widetilde{V} s \right)(p) + \left(\widetilde{V} s \right)(q) \right).$$

Assume E is of rank one. Recall that $s_i^p(y) = \sum_{j=1}^n a(q,p)_{ij}(y) s_j^q(y)$. Hence

$$s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) = \sum_{j=1}^n \left(a(q,p)_{ij}(y) - A(q,p)_{ij} \right) s_j^q(y).$$

Therefore

$$\int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right| dV(y) \leq$$

$$\|s\|_{2,3\varepsilon} \sum_{j=1}^n \left(\int_{B'_{pq}} |a(q,p)_{ij}(y) - A(q,p)_{ij}|^2 \right)^{\frac{1}{2}} dV(y).$$

Finally, as $B'_{pq} \subset B_{pq}$, inequality (A.1) implies

$$\sum_{i=1}^n \left(\int_{B'_{pq}} \left| s_i^p(y) - \sum_{j=1}^n A(q,p)_{ij} s_j^q(y) \right| dV(y) \right)^2 \leq c \left(\widetilde{V} s \right)(p)$$

and this concludes the proof of Lemma 3.22. $\square$

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